Indian Statistical Institute, Bangalore
B. Math (II)
First semester 2011-2012
Semester Examination : Statistics (I)

Date: 30-11-2011 **Maximum Score 75** **Duration: 3 Hours**

1. Let $X_1,\ X_2,\cdots,X_n$ be *independently and identically distributed* (*iid*) with the *probability density function (pdf)* given by

$$f(x|\tau) = (\tau+1)\, x^{\tau} I_{(0,1)}(x)\,, \qquad \tau \geq 0.$$

   (a) Find $\widehat{\tau}_{mle}$, *maximum likelihood estimator (mle)* for τ.

   (b) Also find $\widehat{\tau}_{mom}$, *method of moments (mom) estimator* for τ.

[6 + 4 = 10]

2. The *Cauchy* distribution has been extensively used, among other disciplines, in physics. Let us consider *Cauchy distribution,* $\mathcal{C}(\mu,\sigma)$ with parameters $\mu \in (-\infty,\infty)$ and $\sigma > 0$ having *probability density function (pdf)* given by

$$f(x|\mu,\sigma) = \frac{1}{\pi\sigma\left[1+\left(\frac{(x-\mu)}{\sigma}\right)^2\right]};\ x \in (-\infty,\infty)\,. \qquad (1)$$

   (a) Check that $f(x|\mu,\sigma)$ in (1) is indeed a *pdf.*

   (b) Find the median, mean and mode of the *Cauchy distribution* (1).

   (c) Find the first and third quartiles of the *Cauchy distribution* (1).

   (d) How would you estimate the parameter μ of the *Cauchy distribution* (1) based on a random sample $X_1,\ X_2,\cdots,X_n$, of size n, from (1)?

   (e) If U is uniform on $(0,1)$ then find the distribution of $W = \tan\left[\pi\left(U-\frac{1}{2}\right)\right]$.

   (f) Suppose we can draw observations from uniform distribution on $(0,1)$. *Explain*, using the answer to (e), how you would draw observations from the *Cauchy distribution* $\mathcal{C}(-10,5)$.

[3 + 9 + 6 + 3 + 5 + 4 = 30]

3. Let $(X_1,Y_1),(X_2,Y_2),\cdots,(X_n,Y_n)$ be a random sample from the distribution with following bivariate density function.

$$f_{\rm XY}(x,y) = \frac{1}{cd} I_{(0,d)}(x)\, I_{(x^2,x^2+c)}(y)$$

   where c, d are positive numbers.

   Let $V = \frac{1}{n}\Sigma_{i=1}^n X_i$ and $W = \frac{1}{n}\Sigma_{i=1}^n Y_i$. Find $\rho_{\rm VW}$, the correlation coefficient between V and W. What happens as $c \to 0$?

[12]

[PTO]

4. Derive *likelihood ratio test* (*LRT*) for testing

$$H_0 : \theta = \theta_0 \textit{ versus } H_1 : \theta \neq \theta_0$$

based on a random sample $X_1,\ X_2, \cdots, X_n$, from $\mathcal{N}(\theta, \sigma^2)$, $\theta \in (-\infty, \infty)$ and $\sigma^2 > 0$. Here both the parameters θ and σ^2 are unknown. Take level of significance α to be 10%. Give an expression for the p value for your *LRT*. How do you interpret the p value? Find 90% *confidence lower bound* (*CLB*) for θ. What do you understand by 90% *CLB*?

$[9 + 2 + 2 + 3 + 2 = 18]$

5. Number of defects in a printed circuit board is often found to follow $\mathcal{P}oisson$ distribution. A random sample of 60 printed circuit boards was selected from a production process. Of the selected boards 32 had no defects at all, 15 had one defect each, 9 had 2 defects each and 4 had exactly 3 defects each. Do you think $\mathcal{P}oisson$ distribution is appropriate to model the number of defects in a printed circuit board based on the given data? Substantiate. Find the p value.

[20]